Coefficients of fractional parentage for U(m + p/n + q) ⊃ U(m/n) × U(p/q) and U(m/n) ⊃ U(m) × U(n)

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Coefficients of fractional parentage for $U(m+p/n+q) \supset U(m/n)\times U(p/q)$ and $U(m/n) \supset U(m)\times U(n)$

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Abstract. The outer-product reduction coefficients (ORC) which reduce the representation (rep) induced from the irreps of the permutation groups $S(f_1)$ and $S(f_2)$ into the irreps of $S(f_1+f_2)$ are shown to be the 'indirect coupling' coefficients for the $U(m+p/n+q) \supset U(m/n)\times U(p/q)$ irreducible basis. The non-standard ORC for reducing the rep induced from the non-standard irreps of $S(f_{13}) \supset S(f_1)\times S(f_3)$ and $S(f_{24}) \supset S(f_2)\times S(f_4)$ into that of $S(f) \supset S(f_{12})\times S(f_{34})$, with $f_{ij}=f_i+f_j$, $f=f_{12}+f_{34}$, are identified with the $U(f) \supset U(f_{12})\times U(f_{34})$ Clebsch–Gordan coefficients for the special Gel'fand bases of $U(f_{12})$ and $U(f_{34})$. The $U(m+p/n+q) \supset U(m/n)\times U(p/q)$ CFP, as well as its special case the $U(m+p) \supset U(m)\times U(p)$ CFP, are identified with the $S(f) \supset S(f_{12})\times S(f_{34})$ outer-product isoscalar factor. The $U(m/n) \supset U(m)\times U(n)$ CFP are obtained from the $U(m+n) \supset U(m)\times U(n)$ CFP by simply changing all the partition labels for $U(n)$ into their conjugates (interchanging rows with columns) and taking into account a phase change. The CFP can be calculated from the ORC. Numerical values of the one-body CFP for systems with up to six particles are tabulated.

1. Introduction

There exist many interesting and deep connections between the unitary group $U(n)$ and the permutation group $S(f)$ due to the so called Schur–Weyl duality (Haase and Butler 1984a). A few of them which are related to our discussion here are as follows. The Yamanouchi (YB) or standard basis of $S(f)$ is the special Gel'fand basis of $U(f)$ (Moshinsky 1966); the quasi-standard basis of $S(f)$ is the general Gel'fand basis (GB) of $U(n)$ (Chen *et al* 1977b, Sarma and Saharasbudhe 1980, Chen and Gao 1982); the Clebsch–Gordan coefficients (CGC) of $S(f)$ are the 'indirect coupling' coefficients for the $U(mn) \supset U(m)\times U(n)$ irreducible basis (IRB) (Vanagas 1972, Chen *et al* 1978b); the $S(f) \supset S(f_1)\times S(f_2)$ inner-product isoscalar factors (ISF) are the $U(mn) \supset U(m)\times U(n)$ coefficients of fractional parentage (CFP) (Chen 1981, Chen *et al* 1983d, 1984b).

A study of the $U(m+n) \supset U(m)\times U(n)$ CFP from the Schur–Weyl duality has been carried out along several lines. The $U(m+n) \supset U(m)\times U(n)$ CFP is related to the $9f$ recoupling coefficient of $S(f)$, while the latter is in turn related to the outer-product reduction coefficients (ORC) of $S(f)$ which are the coefficients for reducing the induced rep of $S(f)$ (Kramer 1967). The $6f$ and $9f$ recoupling coefficients of $S(f)$ are identified with the $6f$ and $9f$ recoupling coefficients of $U(n)$ (Kramer 1968, Kramer and Seligman 1969a). The matrix elements of the double coset generators (DCME) of $S(f)$ are identified with the $9f$ recoupling coefficients (Kramer and Seligman 1969b). Later, in a series of papers it has been shown that the DCME of $S(f)$ under the decomposition $\otimes S(f_i)\backslash S(f)/\otimes S(f_j)$ are equal to the DCME of $U(n)$ under the decomposition $\otimes U(n_i)\backslash U(n)/\otimes U(n_j)$ (Sullivan 1975) and the weighted DCME of $S(f)$ are equal to the $U(m+n)\supset U(m)\times U(n)$ CFP (Sullivan 1980b). An iterative procedure for evaluating the DCME is proposed (Sullivan 1980a). The same problem has been attacked by Kaplan (1961a, b) and Kukulin *et al* (1967) by using the transformation coefficients (TC) from the standard to the non-standard basis of $S(f)$ and the $9f$ recoupling coefficients of SU(4), respectively. Thus it can be said that so far the relation between the ORC, TC and recoupling coefficients of $S(f)$ on the one hand, and the $U(m+n)\supset U(m)\times U(n)$ CFP on the other hand, is well established, albeit in a roundabout way. However, a satisfactory algorithm for evaluating the $U(m+n)\supset U(m)\times U(n)$ CFP has not been seen yet, and a systematic tabulation of the CFP is not available.

Section 2 is devoted to re-establishing the relation between the ORC of $S(f)$ and the CFP of $U(n)$ in a most direct way, and to the method of computation of the CFP. Based on Moshinsky's theorem identifying the YB of $S(f)$ with the special GB of $U(f)$, we immediately see that the non-standard ORC (NORC) for reducing the rep induced from the irreps of $S(f_{13})\supset S(f_1)\times S(f_3)$ and $S(f_{24})\supset S(f_2)\times S(f_4)$ to the irreps of $S(f)\supset S(f_{12})\times S(f_{34})$ is nothing else but the $U(f)\supset U(f_{12})\times U(f_{34})$ CGC for the special Gel'fand bases of $U(f_{12})$ and $U(f_{34})$. Then it is trivial to identify the $S(f)\supset S(f_{12})\times S(f_{34})$ outer-product ISF(OISF) with the $U(f)\supset U(f_{12})\times U(f_{34})$ CFP.

With the application of the $U(6/4)\supset U(6)\times U(4)$ IRB to the supersymmetry model in nuclear physics (Iachello 1980, Balantekin *et al* 1981), it is of interest to obtain the $U(m/n)\supset U(m)\times U(n)$ CFP. Since the restriction of the irrep of $U(m/n)$ to irreps of $U(m)\times U(n)$ is very much like that of $U(m+n)$ to $U(m)\times U(n)$ (Dondi and Jarvis 1981, Balentekin 1982), we expect that the $U(m/n)\supset U(m)\times U(n)$ CFP should be very much like the $U(m+n)\supset U(m)\times U(n)$ CFP. The remaining part of this paper is devoted to the CFP for the graded unitary group.

In our previous papers it is shown that the YB of the graded state permutation group $\mathring{\mathscr{S}}(f)$ is the special GB of $U(m/n)$, and the ORC of $\mathring{\mathscr{S}}(f)$ is the CGC for the special GB of $U(m/n)$ (Chen *et al* 1983b, 1984a). In §§ 3 and 4, it will be demonstrated by using the duality argument that the ORC of $S(f)$ is the 'indirect coupling' coefficient for the $U(m+p/n+q)\supset U(m/n)\times U(p/q)$, and its special case, the $U(m/q)\supset U(m)\times U(q)$ IRB, just as the CGC of $S(f)$ is the 'indirect coupling' coefficient for the $U(mn)\supset U(m)\times U(n)$ IRB. In § 5, the $S(f)\supset S(f_{12})\times S(f_{34})$ outer-product ISF is identified with the $U(m+p/n+q)\supset U(m/n)\times U(p/q)$ CFP and $U(m+p)\supset U(m)\times U(p)$ CFP. In § 6, the $U(m/q)\supset U(m)\times U(q)$ CFP is discussed as a special case of the $U(m+p/n+q)\supset U(m/n)\times U(p/q)$ CFP with $n=p=0$. The $U(m/n)\supset U(m)\times U(n)$ CFP is obtained from the $U(m+n)\supset U(m)\times U(n)$ CFP by multiplying a sign factor and changing the partition labels of $U(n)$ into their conjugates. Section 7 contains tables for the $U(m+n)\supset U(m)\times U(n)$ and $U(m/n)\supset U(m)\times U(n)$ one-body CFP calculated from the ORC for systems with up to six particles. These tables are universal in the sense that they are applicable to any m and n, as long as m and n are large enough for the partitions (or Young diagrams) of $U(m)$ and $U(n)$ to be permissible. Section 8 is a summary and discussion on the duality of the reduction coefficients of $S(f)$ and $U(m/n)$ as well as on the phase problem.

2. The non-standard ORC and $U(m+n) \supset U(m) \times U(n)$ CGC

The ORC first appeared in Kramer (1967). An improved projection operator method for calculating the ORC was suggested by William and Pursey (1976). A systematic study of the ORC, including definition, symmetry properties, applications, algorithm and tabulation, was undertaken by Chen *et al* (1978a) and Chen and Gao (1981). A more recent study of the induction transformation is given by Haase and Butler (1984b).

Let

$$(\bar{\omega}_1) = (12, \ldots, f_1), \qquad (\bar{\omega}_2) = (f_1+1, \ldots, f), \qquad f = f_1 + f_2, \tag{2.1a}$$

and $S(f_i)$ be the permutation group operating on the numbers of $(\bar{\omega}_i)$, $i = 1, 2$. Next we introduce $\binom{f}{f_1} = f!/f_1!f_2!$ ordered sets $(\omega) = (\omega_1, \omega_2)$,

$$\begin{aligned} (\omega_1) &= (a_1, a_2, \ldots, a_{f_1}), \qquad && a_1 < a_2 < \ldots < a_{f_1}, \\ (\omega_2) &= (a_{f_1+1}, \ldots, a_f), && a_{f_1+1} < \ldots < a_f, \end{aligned} \tag{2.1b}$$

constructed out of the numbers $1, 2, \ldots, f$. The left coset decomposition of $S(f)$ with respect to the subgroup $S(f_1) \times S(f_2)$ is denoted by

$$S(f) = \sum_{\omega} \oplus Q_\omega (S(f_1) \times S(f_2)), \tag{2.2}$$

where the left coset representatives Q_ω are just the so-called order-preserving permutations (MacFarlane and French 1960),

$$Q_\omega = \begin{pmatrix} \bar{\omega} \\ \omega \end{pmatrix}, \qquad (\bar{\omega}) = (1, 2, \ldots, f). \tag{2.3}$$

Applying the $\binom{f}{f_1}$ Q_ω's to the IRB of $S(f_1) \times S(f_2)$,

$$|Y_{r_1}^{\sigma_1}(\bar{\omega}_1)\rangle |Y_{r_2}^{\sigma_2}(\bar{\omega}_2)\rangle, \qquad r_i = 1, 2, \ldots, \dim(\sigma_i), \tag{2.4a}$$

σ_i and r_i being the partitions and Yamanouchi symbols, while $\dim(\sigma_i)$ are the dimensions of the irreps $[\sigma_i]$ of $S(f_i)$, we get altogether $\dim(\sigma_1)\dim(\sigma_2)\binom{f}{f_1}$ basis vectors

$$|Y_{r_1}^{\sigma_1}(\omega_1)\rangle |Y_{r_2}^{\sigma_2}(\omega_2)\rangle, \tag{2.4b}$$

where $Y_{r_i}^{\sigma_i}(\omega_i)$ denotes a generalised Young tableau formed by filling the Young diagram $[\sigma_i]$ with the numbers (ω_i) according to an order specified by the Yamanouchi symbol r_i.

The basis vectors (2.4*b*) carry the induced rep of $S(f)$, which can be reduced into irreps of $S(f)$ through the use of the ORC,

$$\begin{aligned} &|\sigma m\rangle \equiv |Y_r^\sigma(\bar{\omega})\rangle = \sum_{m_1 m_2} \langle \sigma\theta m | \sigma_1 m_1 \sigma_2 m_2\rangle |\sigma_1 m_1\rangle |\sigma_2 m_2\rangle, \\ &m = r\bar{\omega} \equiv r, \qquad |\sigma_i m_i\rangle = |Y_{r_i}^{\sigma_i}(\omega_i)\rangle, \qquad m_i = r_i\omega_i, \end{aligned} \tag{2.5a}$$

where

$$\langle \sigma\theta m | \sigma_1 m_1 \sigma_2 m_2\rangle \equiv \langle \sigma\theta r | \sigma_1 r_1 \omega_1 \sigma_2 r_2 \omega_2\rangle \tag{2.5b}$$

is the ORC, θ being the multiplicity label. Notice that the summation over m_1 and m_2 in (2.5) is equivalent to that over r_1, r_2 and ω_1 (or ω_2). The same remark applies to all similar cases below. The ORC satisfy the unitarity conditions

$$\sum_{m_1m_2} \langle\sigma\theta m|\sigma_1m_1\sigma_2m_2\rangle\langle\sigma'\theta'm'|\sigma_1m_1\sigma_2m_2\rangle = \delta_{\sigma\sigma'}\delta_{\theta\theta'}\delta_{mm'},$$
$$\sum_{\sigma\theta r} \langle\sigma\theta m|\sigma_1m_1\sigma_2m_2\rangle\langle\sigma\theta m|\sigma_1m_1'\sigma_2m_2'\rangle = \delta_{m_1m_1'}\delta_{m_2m_2'}. \quad (2.6)$$

Since a Yamanouchi basis vector of $S(f)$ is a simultaneous eigenfunction of the $f-1$ two-cycle class operators $C_{(2)}(n)$ of $S(n)$, $n=f, f-1, \ldots, 2$ (Chen *et al* 1977a, Chen and Gao 1982), the ORC can be calculated by diagonalising the $f-1$ two-cycle class operators in the basis (2.4*b*). Based on this algorithm, a code in ALGOL-60 has been written and the ORC of S(2)–S(6) have been tabulated (Chen and Gao 1981).

Moshinsky (1966) has proved that the YB of $S(f)$ under the Young–Yamanouchi (YY) phase convention is identical (including the phase) to the special GB of $U(f)$ under the Gel'fand–Zetlin (GZ) phase convention. Therefore, the generalised Young tableau in (2.4*b*) can be regarded as the Weyl tableau; $|\sigma m\rangle$ and $|\sigma_i m_i\rangle$ are the special GB of $U(f)$; whereas the ORC in (2.5) are just the CGC for the special GB of $U(f)$ (Chen *et al* 1978a).

Kramer (1967) used the notation $\langle\theta\sigma r|\sigma_1r_1\sigma_2r_2, \omega\rangle$ for the ORC. We prefer to use $\langle\sigma\theta m|\sigma_1m_1\sigma_2m_2\rangle$, since it reminds us that the ORC is a CGC of the $U(f)$ special GB.

To be concise in notation, we will drop all the multiplicity labels in most of our discussion below, and restore them only when it is necessary. We will also stick to the YY and GZ phase conventions.

Now we extend the ORC to the non-standard ORC. Let

$$f=f_1+f_2+f_3+f_4, \qquad f_{ij}=f_i+f_j.$$

Group the numbers $1, 2, \ldots, f$ into four sets (ω_i) consisting of f_i numbers in ascending order under the restriction that

$$(\omega_1), (\omega_2) \in (1, 2, \ldots, f_{12}), \qquad (\omega_3), (\omega_4) \in (f_{12}+1, \ldots, f). \quad (2.7)$$

There are altogether $\binom{f_{12}}{f_1}$ ordered sets of (ω_1, ω_2) and $\binom{f_{34}}{f_3}$ ordered sets of (ω_3, ω_4). The NORC are the coefficients for reducing the rep induced from the irreps of $S(f_{13}) \supset S(f_1)\times S(f_3)$ and $S(f_{24}) \supset S(f_2)\times S(f_4)$ into the irreps of $S(f) \supset S(f_{12})\times S(f_{34})$, namely the coefficients in the following expansion:

$$\left|\begin{matrix}\sigma\\ \sigma_{12}m_{12}\sigma_{34}m_{34}\end{matrix}\right\rangle \equiv \left|\begin{matrix}\sigma\\ \sigma_{12}r_{12}\sigma_{34}r_{34}\end{matrix}\right\rangle$$
$$= \sum_{\substack{\sigma_1\ldots\sigma_4\\ m_1\ldots m_4}} \left\langle\begin{matrix}\sigma\\ \sigma_{12}m_{12}\sigma_{34}m_{34}\end{matrix}\middle|\begin{matrix}\sigma_{13} & \sigma_{24}\\ \sigma_1m_1\sigma_3m_3, & \sigma_2m_2\sigma_4m_4\end{matrix}\right\rangle$$
$$\times\left|\begin{matrix}\sigma_{13}\\ \sigma_1m_1\sigma_3m_3\end{matrix}\right\rangle\left|\begin{matrix}\sigma_{24}\\ \sigma_2m_2\sigma_4m_4\end{matrix}\right\rangle, \quad (2.8)$$

where $m_i = r_i\omega_i$, and $\left|{}^{\sigma_{13}}_{\sigma_1m_1\sigma_3m_3}\right\rangle$ is the $S'(f_{13}) \supset S'(f_1)\times S'(f_3)$ IRB, $S'(f_{13})$, $S'(f_1)$ and $S'(f_3)$ being the permutation groups operating on (ω_1, ω_3), (ω_1) and (ω_3), respectively. The meaning of the other two IRB in (2.8) is similar to the above.

According to Moshinsky's theorem and the assigned ranges for the numbers in the sets (ω_i) specified by (2.7), it is readily seen that the YB $|\sigma_1m_1\rangle$, $|\sigma_2m_2\rangle$ and $|\sigma_{12}m_{12}\rangle$ are the special GB of $U(f_{12})$; $|\sigma_3m_3\rangle$, $|\sigma_4m_4\rangle$ and $|\sigma_{34}m_{34}\rangle$ are the special GB of $U(f_{34})$;

whereas all the three non-standard bases of the permutation groups $S(f)$, $S'(f_{13})$ and $S'(f_{24})$ in (2.8) are the $U(f) \supset U(f_{12}) \times U(f_{34})$ IRB, with the unitary groups $U(f)$, $U(f_{12})$ and $U(f_{34})$ operating on the indices $(1, 2, \ldots, f)$, $(1, 2, \ldots, f_{12})$ and $(f_{12}+1, \ldots, f)$ respectively. This means that the NORC in (2.8) are identical (including the phase) to the $U(f) \supset U(f_{12}) \times U(f_{34})$ CGC for the special GB of $U(f_{12})$ and $U(f_{34})$.

We note that an equation similar to (2.8) is given by Kramer (1967). However, his equation (4.15) is incorrect. The point is that $\sigma_1, \ldots, \sigma_4$ are the summation indices instead of the arguments of the non-standard basis of $S(f)$.

The NORC can be factorised as

$$\left\langle \begin{matrix} \sigma \\ \sigma_{12}m_{12}\sigma_{34}m_{34} \end{matrix} \middle| \begin{matrix} \sigma_{13} & \sigma_{24} \\ \sigma_1 m_1 \sigma_3 m_3, & \sigma_2 m_2 \sigma_4 m_4 \end{matrix} \right\rangle$$

$$= \left\langle \begin{matrix} \sigma \\ \sigma_{12}\sigma_{34} \end{matrix} \middle| \begin{matrix} \sigma_{13} & \sigma_{24} \\ \sigma_1\sigma_3, & \sigma_2\sigma_4 \end{matrix} \right\rangle \langle \sigma_{12}m_{12} | \sigma_1 m_1 \sigma_2 m_2 \rangle \langle \sigma_{34}m_{34} | \sigma_3 m_3 \sigma_4 m_4 \rangle, \quad (2.9a)$$

namely,

$$\text{NORC} = (S(f) \supset S(f_{12}) \times S(f_{34}) \text{ OISF}) \times S(f_{12}) \text{ ORC} \times S(f_{34}) \text{ ORC}.$$

From the unitary group point of view, the same equation can be interpreted as

$$\text{CGC} = (U(f) \supset U(f_{12}) \times U(f_{34}) \text{ CFP}) \times U(f_{12}) \text{ CGC} \times U(f_{34}) \text{ CGC}.$$

It follows immediately that the $S(f) \supset S(f_{12}) \times S(f_{34})$ OISF is precisely the $U(f) \supset U(f_{12}) \times U(f_{34})$ CFP. To stress this point, let us write

$$\left\langle \begin{matrix} \sigma \\ \sigma_{12}\sigma_{34} \end{matrix} \middle| \begin{matrix} \sigma_{13} & \sigma_{24} \\ \sigma_1\sigma_3, & \sigma_2\sigma_4 \end{matrix} \right\rangle_{S(f)} = \left\langle \begin{matrix} \sigma \\ \sigma_{12}\sigma_{34} \end{matrix} \middle| \begin{matrix} \sigma_{13} & \sigma_{24} \\ \sigma_1\sigma_3, & \sigma_2\sigma_4 \end{matrix} \right\rangle_{U(m+n)}. \quad (2.10)$$

Upon using (2.9a) and (2.6), the CFP can be expressed as

$$\left\langle \begin{matrix} \sigma \\ \sigma_{12}\sigma_{34} \end{matrix} \middle| \begin{matrix} \sigma_{13} & \sigma_{24} \\ \sigma_1\sigma_3, & \sigma_2\sigma_4 \end{matrix} \right\rangle = \sum_{m_1,\ldots,m_4} \left\langle \begin{matrix} \sigma \\ \sigma_{12}m_{12}\sigma_{34}m_{34} \end{matrix} \middle| \begin{matrix} \sigma_{13} & \sigma_{24} \\ \sigma_1 m_1 \sigma_3 m_3, & \sigma_2 m_2 \sigma_4 m_4 \end{matrix} \right\rangle$$

$$\times \langle \sigma_{12}m_{12} | \sigma_1 m_1 \sigma_2 m_2 \rangle \langle \sigma_{34}m_{34} | \sigma_3 m_3 \sigma_4 m_4 \rangle. \quad (2.9b)$$

Kramer (1967) shows that (2.9b) can be expressed in terms of the $9f$ recoupling coefficient of $S(f)$,

$$\left\langle \begin{matrix} \sigma \\ \sigma_{12}\sigma_{34} \end{matrix} \middle| \begin{matrix} \sigma_{13} & \sigma_{24} \\ \sigma_1\sigma_3, & \sigma_2\sigma_4 \end{matrix} \right\rangle = C^{-1}(\sigma_1\sigma_2\sigma_{12}) C^{-1}(\sigma_3\sigma_4\sigma_{34}) C(\sigma_{13}\sigma_{24}\sigma) \begin{bmatrix} \sigma_1 & \sigma_2 & \sigma_{12} \\ \sigma_3 & \sigma_4 & \sigma_{34} \\ \sigma_{13} & \sigma_{24} & \sigma \end{bmatrix}, \quad (2.11a)$$

where

$$C(\sigma_1\sigma_2\sigma_{12}) = \left(\frac{\dim(\sigma_{12})}{\dim(\sigma_1)\dim(\sigma_2)} \frac{f_1! f_2!}{f_{12}!} \right)^{1/2}. \quad (2.11b)$$

By using the TC of the permutation groups (Jahn 1954,Kaplan 1961a, b, Kramer 1967), and noting that the summation over $m_1, \ldots, m_4$ is equivalent to that over $r_1, \ldots, r_4$, ω_1 and ω_3, and the summation over r_{13} and r_{24} implies a summation over

r_1 and r_2, the CFP in (2.9b) can be put into the form

$$\left\langle \begin{matrix} \sigma \\ \sigma_{12}\sigma_{34} \end{matrix} \right| \left. \begin{matrix} \sigma_{13} & \sigma_{24} \\ \sigma_1\sigma_3, & \sigma_2\sigma_4 \end{matrix} \right\rangle^{\theta_{13}\theta_{24}\theta_{1324}}_{\theta_{12}\theta_{34}\theta_{1234}} = \sum_{r_{13}r_{24}r_3r_4r}^{\text{fix } r_{12}r_{34}} \sum_{\omega_1\omega_3} \langle \sigma r | \theta_{1234}, \sigma_{12}r_{12}\sigma_{34}r_{34}\rangle \langle \sigma_{13}r_{13} | \theta_{13}, \sigma_1 r_1 \sigma_3 r_3\rangle$$
$$\times \langle \sigma_{24}r_{24} | \theta_{24}, \sigma_2 r_2 \sigma_4 r_4\rangle \langle \sigma\theta_{1324} m | \sigma_{13}m_{13}\sigma_{24}m_{24}\rangle$$
$$\times \langle \sigma_{12}\theta_{12}m_{12} | \sigma_1 m_1 \sigma_2 m_2\rangle \langle \sigma_{34}\theta_{34}m_{34} | \sigma_3 m_3 \sigma_4 m_4\rangle, \qquad (2.12)$$

where the first three factors on the RHS are the TC, and

$$m_{ij} = r_{ij}\omega_{ij}, \qquad (\omega_{ij}) = (\omega_i, \omega_j), \qquad \text{for } (ij) = (13), (24).$$

With the programs or tables for the TC (Chen *et al* 1983c), and ORC (Chen and Gao 1981), from (2.12) we are able to evaluate the $\mathrm{U}(m+n) \supset \mathrm{U}(m) \times \mathrm{U}(n)$ CFP.

In the case where any one of the multiplicity labels is redundant, or when only the one- or two-body CFP is concerned, simplifications occur for (2.12). For example, for the one-body CFP, $f_{34} = \sigma_{34} = 1$. Then either $\sigma_3 = 1, \sigma_4 = 0$ or $\sigma_3 = 0, \sigma_4 = 1$. Therefore the indices σ_3, σ_4, σ_{34}, θ_{13}, θ_{24} and θ_{1234} are redundant, and (2.12) reduces to the $\mathrm{S}(f) \supset \mathrm{S}(f-1)$ OISF

$$\left\langle \begin{matrix} \sigma \\ \sigma_{12}\sigma_{34} \end{matrix} \right| \left. \begin{matrix} \sigma_{13} & \sigma_{24} \\ \sigma_1\sigma_3, & \sigma_2\sigma_4 \end{matrix} \right\rangle^{\theta_{13}\theta_{24}\theta_{1324}}_{\theta_{12}\theta_{34}\theta_{1234}} \to \left\langle \begin{matrix} \sigma \\ \sigma_{12} \end{matrix} \right| \left. \begin{matrix} \sigma_{13} & \sigma_{24} \\ \sigma_1 & \sigma_2 \end{matrix} \right\rangle^{\theta_{1324}}_{\theta_{12}}$$
$$\to \left\langle \begin{matrix} \sigma \\ \sigma' \end{matrix} \right| \left. \begin{matrix} \sigma_1 & \sigma_2 \\ \sigma_1' & \sigma_2' \end{matrix} \right\rangle^{\theta}_{\theta'} = \sum_{m_1'm_2'} \langle \sigma\theta m | \sigma_1 m_1 \sigma_2 m_2\rangle \langle \sigma'\theta' m' | \sigma_1' m_1' \sigma_2' m_2'\rangle, \qquad (2.13a)$$

where simpler notations for the partition labels have been used; the two ORC in (2.13a) refer to the $\mathrm{S}(f)$ and $\mathrm{S}(f-1)$ groups. Note that $|\sigma' m'\rangle$ and $|\sigma_i' m_i'\rangle$ are the basis vectors resulting from ignoring the last particle f in the basis vectors $|\sigma m\rangle$ and $|\sigma_i m_i\rangle$, $i = 1, 2$, respectively, and due to the branching rule, we have

$$[\sigma]m = [\sigma][\sigma']m', \qquad [\sigma_i]m_i = [\sigma_i][\sigma_i']m_i'. \qquad (2.13b)$$

Obviously, if the last particle is not in $|\sigma_i m_i\rangle$, then $|\sigma_i' m_i'\rangle = |\sigma_i m_i\rangle$.

When the multiplicity label θ' is redundant, (2.13a) is further reduced to

$$\left\langle \begin{matrix} \sigma \\ \sigma' \end{matrix} \right| \left. \begin{matrix} \sigma_1 & \sigma_2 \\ \sigma_1' & \sigma_2' \end{matrix} \right\rangle^{\theta} = \langle \sigma\theta m | \sigma_1 m_1 \sigma_2 m_2\rangle / \langle \sigma' m' | \sigma_1' m_1' \sigma_2' m_2'\rangle. \qquad (2.13c)$$

For example,

$$\left\langle \begin{matrix} [321] \\ [311] \end{matrix} \right| \left. \begin{matrix} [21] & [21] \\ [2] & [21] \end{matrix} \right\rangle^{(\theta)} =$$
$$\left\langle {}^{\theta}\begin{smallmatrix} 1 & 3 & 4 \\ 2 & 6 & \\ 5 & & \end{smallmatrix} \right| \left. \begin{smallmatrix} 1 & 3 \\ 6 & \end{smallmatrix} \quad \begin{smallmatrix} 2 & 4 \\ 5 & \end{smallmatrix} \right\rangle \Bigg/ \left\langle \begin{smallmatrix} 1 & 3 & 4 \\ 2 & & \\ 5 & & \end{smallmatrix} \right| \left. \begin{smallmatrix} 1 & 3 \end{smallmatrix} \quad \begin{smallmatrix} 2 & 4 \\ 5 & \end{smallmatrix} \right\rangle$$
$$= \begin{cases} (\frac{1}{80})^{1/2} / (\frac{3}{20})^{1/2} = (\frac{1}{12})^{1/2}, & \text{for } \theta = 1, \\ (\frac{1}{16})^{1/2} / (\frac{3}{20})^{1/2} = (\frac{5}{12})^{1/2}, & \text{for } \theta = 2, \end{cases} \qquad (2.14)$$

where the S(6) and S(5) ORC are taken from Chen and Gao (1981).

From this example we can see that it is trivial to obtain the one-body CFP once the ORC are known. The $U(m+n) \supset U(m) \times U(n)$ one-body CFP have been calculated from (2.13) and are tabulated in § 7.

3. The $\mathbf{U(m+p/n+q) \supset U(m/n) \times U(p/q)}$ IRB

For establishing the duality of the reduction coefficients of $\mathring{S}(f)$ and $U(m/n)$, we need the following two preliminary theorems.

Theorem 1. The necessary and sufficient condition for a function $\psi^{(\sigma)}$ to belong to the basis space of an irrep (σ) of a group G is that $\psi^{(\sigma)}$ is an eigenfunction of the CSCO of G possessing those eigenvalues which serve to characterise (σ) (Chen *et al* 1977a, 1983e).

The CSCO of G is a complete set of commuting operators in the class space of G. For a compact Lie group of rank l, the CSCO is just the set of l Casimir invariants of G, while for a finite group, the CSCO consists of a few class operators of G. For example, the CSCO of the graded permutation group $\mathring{S}(f)$ is $\mathring{C}(f) = (\mathring{C}_{(2)}(f), \mathring{C}_{(3)}(f))$ for $f \leqslant 14$, where $\mathring{C}_{(i)}(f)$ is the i-cycle class operator of $\mathring{S}(f)$ (Chen *et al* 1983b).

Theorem 2. In the f-particle product-state space with $M(N)$ boson (fermion) single-particle states, referred to as the graded space L, the Casimir invariants $I_k^{M/N}$ of $U(M/N)$ are functions of the CSCO of $\mathring{S}(f)$ (Chen *et al* 1983b),

$$I_k^{M/N} = F_k^{M/N}(\mathring{C}(f)), \qquad k = 1, 2, \ldots, M+N. \tag{3.1}$$

As we will see, it is due to these two important theorems that each reduction coefficient of $\mathring{S}(f)$ of $U(M/N)$ plays dual roles; one is related to its own group, called the direct role, and the other is related to another group, called the indirect role.

Consider two subsystems, one in the configuration $A^{f_1}B^{f_2}$, and the other in $A^{f_3}B^{f_4}$, $A = A_1, A_2, \ldots, A_{m+n}$, $B = B_1, B_2, \ldots, B_{p+q}$, spanning the defining rep of $U(m/n)$ and $U(p/q)$, respectively. In the graded space L, we need two types of labels to specify a basis vector uniquely, one specifying its transformation under the graded permutation group and the other specifying its transformation under the graded unitary group. Let

$$\left|\begin{matrix}\sigma_i \\ m_i w_i\end{matrix}\right\rangle^{\circ} = \left|\begin{matrix}\sigma_i \\ r_i\omega_i,\ w_i\end{matrix}\right\rangle^{\circ}, \qquad \begin{aligned} r_i &= 1, 2, \ldots, \dim(\sigma_i), \\ w_i &= 1, 2, \ldots, \mathrm{Dim}(\sigma_i), \end{aligned} \tag{3.2}$$

be the YB of $\mathring{S}(f_i)$ operating on the indices (ω_i) and the IRB of $U(m/n)$ for $i = 1, 3$ and of $U(p/q)$ for $i = 2, 4$, where the meaning of (ω_i) is the same as in § 2, and $\mathrm{Dim}(\sigma_i)$ is the dimension of the irrep $[\sigma_i]$ of $U(m/n)$ or $U(p/q)$.

Theorem 3. The ORC of $S(f_{12})$ are the 'indirect coupling' coefficients for the $U(m+p/n+q) \supset U(m/n) \times U(p/q)$ IRB:

$$\left|\begin{matrix}\sigma_{12} \\ m_{12}, \sigma_1 w_1 \sigma_2 w_2\end{matrix}\right\rangle^{\circ} = \sum_{m_1 m_2} \langle \sigma_{12} m_{12} | \sigma_1 m_1 \sigma_2 m_2 \rangle \left|\begin{matrix}\sigma_1 \\ m_1 w_1\end{matrix}\right\rangle^{\circ} \left|\begin{matrix}\sigma_2 \\ m_2 w_2\end{matrix}\right\rangle^{\circ}. \tag{3.3}$$

Proof. Since the graded permutation group $\mathring{S}(f)$ and the ordinary permutation group $S(f)$ are isomorphic (Dondi and Jarvis 1981), they must have identical irreducible matrix elements, CGC, ORC and TC. Therefore the LHS of (3.3) is the YB of $\mathring{S}(f_{12})$. According to theorem 1, it has to be an eigenfunction of the CSCO of $\mathring{S}(f_{12})$. On the other hand, due to (3.1), it is necessarily an eigenfunction of the CSCO of $U(m+p/n+q)$. Again by theorem 1, it must belong to an irrep of $U(m+p/n+q)$, which can be conveniently labelled again by the partition $[\sigma_{12}]$. Besides, the indices $\sigma_1 w_1$ and $\sigma_2 w_2$ are kept fixed in the summation in (3.3), therefore the LHS of (3.3) continues to be the IRB of $U(m/n)$ and $U(p/q)$. All taken together, equation (3.3) is the YB of $\mathring{S}(f_{12})$ and the $U(m+p/n+q) \supset U(m/n) \times U(p/q)$ IRB.

From the above discussion we see that due to theorems 1 and 2, in reducing the induced rep of $\mathring{S}(f_{12})$, the ORC 'indirectly' couple the IRB of $\mathring{S}(f_1)$ and $U(m/n)$ with the IRB of $\mathring{S}(f_2)$ and $U(p/q)$ to that of $U(m+p/n+q)$.

When $n=q=0$, the $U(m+p/n+q) \supset U(m/n) \times U(p/q)$ IRB (3.3) reduces to the $U(m+p) \supset U(m) \times U(p)$ IRB.

By interchanging the roles played by the graded permutation group and the graded unitary group, and following the same reasoning, we have:

Theorem 4. The CGC of $U(m/n)$ are the 'indirect coupling' coefficients for the $\mathring{S}(f_{13}) \supset \mathring{S}(f_1) \times \mathring{S}(f_3)$ IRB,

$$\left|\begin{matrix} \sigma_{13} \\ \sigma_1 m_1 \sigma_3 m_3, \; w_{13} \end{matrix}\right\rangle^{\circ} = \sum_{w_1 w_3} {}^{\circ}\langle \sigma_{13} w_{13} | \sigma_1 w_1 \sigma_3 w_3 \rangle^{\circ} \left|\begin{matrix} \sigma_1 \\ m_1 w_1 \end{matrix}\right\rangle^{\circ} \left|\begin{matrix} \sigma_3 \\ m_3 w_3 \end{matrix}\right\rangle^{\circ}. \qquad (3.4)$$

where ${}^{\circ}\langle \sigma_{12} w_{13} | \sigma_1 w_1 \sigma_3 w_3 \rangle^{\circ}$ designate the CGC of $U(m/n)$.

When $n=0$, (3.4) reduces to equation (6.8) in Kramer (1968).

4. The $U(m/n) \supset U(m) \times U(n)$ IRB

When $n=p=0$, (3.3) becomes the $U(m/q) \supset U(m) \times U(0/q)$ IRB with the $U(m/0) (=U(m))$ IRB $|{}^{\sigma_1}_{m_1 w_1}\rangle^{\circ} = |{}^{\sigma_1}_{m_1 w_1}\rangle$ referring to bosons and the $U(0/q)$ IRB $|{}^{\sigma_2}_{m_2 w_2}\rangle^{\circ}$ to fermions. We will show that the $U(m/q) \supset U(m) \times U(0/q)$ IRB differs from the $U(m/q) \supset U(m) \times U(q)$ IRB only in a phase factor. To this end, we need to find the relationship between the IRB of $U(0/q)$ and $U(q)$.

Up to now, we have had no restriction whatsoever on the basis choice of $U(m/n)$ and $U(p/q)$. For definiteness, in what follows we will choose the extended GB for them and thus the index w represents a graded Weyl tableau (Chen *et al* 1983b), keeping in mind that the discussion is valid for other choices of basis as well, since the CFP is basis independent.

According to Chen *et al* (1983b), the Gel'fand basis of $U(p/q)$ and the YB of $\mathring{S}(f)$ can be constructed by applying a non-standard projection operator $\mathring{P}_r^{[\sigma](m)}$ of $\mathring{S}(f)$ to the f-particle product state $|\Phi\rangle$. After adjusting to the notation used here, equation (2.27) in Chen *et al* (1983b) reads

$$\left|\begin{matrix} [\sigma] \\ r, \; w \end{matrix}\right\rangle^{\circ} = \left|\begin{matrix} [\sigma] \\ r, \; (m) \end{matrix}\right\rangle^{\circ} = \mathring{P}_r^{[\nu](m)} |\Phi\rangle, \qquad (4.1a)$$

$$\mathring{P}_r^{[\sigma](m)} = \left(\frac{h_\sigma}{f!}\right)^{1/2} \sum_p \langle [\sigma] r | p | [\sigma](m) \rangle \mathring{p}, \qquad (4.1b)$$

where $|[\sigma](m)\rangle$ is a non-standard basis vector of $\mathring{S}(f)$ which has a one-to-one correspondence with the graded Weyl tableau w. For a totally bosonic tableau w (i.e. the ordinary Weyl tableau), $|[\sigma](m)\rangle$ is represented by the so-called symmetric parenthesis (essentially the extended Yamanouchi symbol) introduced by Sarma and Saharasbudhe (1980), whereas for a totally fermionic w, $|[\sigma](m)\rangle$ is represented by the antisymmetric parenthesis (Chen and Chen 1983). The non-standard basis $|[\sigma](m)\rangle$ can be expanded in terms of the YB of $S(f)$. Suppose $|[\sigma](m)\rangle$ is an antisymmetric parenthesis,

$$|[\sigma](m)\rangle = \sum_s a_{(m),s}|[\sigma]s\rangle. \tag{4.2}$$

According to the phase convention (Chen and Chen 1983), the coefficient $a_{(m),s_0}$ associated with the maximum possible (in Hamermesh's ordering (Hamermesh 1962)) Yamanouchi symbol s_0 is chosen to be positive,

$$a_{(m),s_0} > 0. \tag{4.3}$$

From the symmetry of the expansion coefficients under conjugation (Chen and Chen 1983, § 3), the symmetric parenthesis $|[\tilde{\sigma}](\tilde{m})\rangle$ conjugated to the antisymmetric parenthesis $|[\sigma](m)\rangle$ is easily obtained from (4.2) by changing the YB $|[\sigma]s\rangle$ to its conjugate and inserting the phase factor Λ_s (Hamermesh 1962, p 266) in front of $|[\tilde{\sigma}]\tilde{s}\rangle$, i.e.

$$|[\tilde{\sigma}](\tilde{m})\rangle = \sum_s a_{(\tilde{m}),\tilde{s}}|[\tilde{\sigma}]\tilde{s}\rangle = \Lambda_{(m)} \sum_s a_{(m),s}\Lambda_s|[\tilde{\sigma}]\tilde{s}\rangle, \tag{4.4a}$$

where the sign factor $\Lambda_{(m)}$ is decided by the phase rule that the coefficients $a_{(\tilde{m}),\tilde{s}}$ for the symmetric parenthesis are always positive (Chen and Chen 1983). Choosing $\tilde{s}$ to be the conjugate of the maximum possible Yamanouchi symbol s_0, from (4.4a) we have

$$a_{(\tilde{m}),\tilde{s}_0} = \Lambda_{(m)} a_{(m),s_0} \Lambda_{s_0} > 0.$$

Due to (4.3) we get

$$\Lambda_w \equiv \Lambda_{(m)} = \Lambda_{s_0}. \tag{4.4b}$$

Assuming $|\Phi\rangle$ is totally fermionic,

$$\mathring{p}|\Phi\rangle = p\delta_p|\Phi\rangle, \tag{4.5}$$

where δ_p is the parity of the permutation p. Inserting (4.5) into (4.1), using the property of the YY matrix element

$$\langle[\sigma]r|p|[\sigma]s\rangle\delta_p = \Lambda_r\Lambda_s\langle[\tilde{\sigma}]\tilde{r}|p|[\tilde{\sigma}]\tilde{s}\rangle, \tag{4.6}$$

and (4.4), we have

$$\left|\begin{matrix}[\sigma] \\ r, w\end{matrix}\right\rangle^{\circ} = \Lambda_r\Lambda_w \left|\begin{matrix}[\tilde{\sigma}] \\ \tilde{r}, \tilde{w}\end{matrix}\right\rangle \tag{4.7a}$$

where

$$\left|\begin{matrix}[\tilde{\sigma}] \\ \tilde{r}, \tilde{w}\end{matrix}\right\rangle = P_{\tilde{r}}^{[\tilde{\sigma}](\tilde{m})}|\Phi\rangle \tag{4.7b}$$

is the IRB of $U(q)$ and the ordinary permutation group $S(f)$, and $\tilde{w}$ the ordinary Weyl tableau conjugate (interchanging rows with columns) to the graded Weyl tableau w. Equation (4.7a) is the extension of (31b) in Chen *et al* (1983b).

Using (4.7a) and interchanging $\sigma_2 \leftrightarrow \tilde{\sigma}_2$, $r_2 \leftrightarrow \tilde{r}_2$ and $w_2 \leftrightarrow \tilde{w}_2$, the $U(m/q) \supset U(m) \times U(0/q)$ IRB (3.3) reads

$$\left|\begin{matrix}\sigma_{12} \\ m_{12}, \sigma_1 w_1 \tilde{\sigma}_2 \tilde{w}_2\end{matrix}\right\rangle^{\circ} = \Lambda_{\tilde{w}_2} \sum_{m_1 m_2} \langle\sigma_{12}m_{12}|\sigma_1 m_1 \tilde{\sigma}_2 \tilde{m}_2\rangle \Lambda_{\tilde{r}_2} \left|\begin{matrix}\sigma_1 \\ m_1 w_1\end{matrix}\right\rangle \left|\begin{matrix}\sigma_2 \\ m_2 w_2\end{matrix}\right\rangle, \tag{4.8}$$

where $\tilde{m}_2 = \tilde{r}_2\omega_2$. It is seen that the IRB of $U(m/q) \supset U(m) \times U(0/q)$ and $U(m/q) \supset U(m) \times U(q)$ only differ by the phase factor $\Lambda_{\tilde{w}_2}$. Therefore the $U(m/q) \supset U(m) \times$

U(q) IRB can be expressed by

$$\left|\begin{matrix}\mathring{S}(f_{12}) & U(m/q) & U(m) & U(q)\\ \sigma_{12}r_{12}; & \sigma_{12}; & \sigma_1 w_1; & \sigma_2 w_2\end{matrix}\right\rangle$$

$$\equiv\left|\begin{matrix}\mathring{\sigma}_{12}\\ \mathring{m}_{12},\ \sigma_1 w_1\sigma_2 w_2\end{matrix}\right\rangle=\Lambda_{\tilde{w}_2}\left|\begin{matrix}\sigma_{12}\\ m_{12},\ \sigma_1 w_1\tilde{\sigma}_2\tilde{w}_2\end{matrix}\right\rangle^{\circ}$$

$$=\sum_{m_1 m_2}\langle\sigma_{12}m_{12}|\sigma_1 m_1\tilde{\sigma}_2\tilde{m}_2\rangle\Lambda_{\tilde{r}_2}\left|\begin{matrix}\sigma_1\\ m_1 w_1\end{matrix}\right\rangle\left|\begin{matrix}\sigma_2\\ m_2 w_2\end{matrix}\right\rangle. \tag{4.9}$$

Example. Let us give the simplest non-trivial example to illustrate the use of the formula (4.9), namely for constructing the YB basis of the graded permutation group $\mathring{S}(4)$ and the $U(1/2)\supset U(1)\times U(2)$ IRB

$$\left|\begin{matrix}\mathring{S}(4); & U(1/2); & U(1); & U(2)\\ [31]3 & [31] & \boxed{a} & \begin{smallmatrix}\alpha & \alpha\\ \beta & \end{smallmatrix}\end{matrix}\right\rangle=\left|\ \begin{smallmatrix}1 & \mathring{3} & 4\\ 2 & & \end{smallmatrix},\ \boxed{a}\ \begin{smallmatrix}\alpha & \alpha\\ \beta & \end{smallmatrix}\right\rangle \tag{4.10}$$

with a denoting a bosonic state, and α, β two fermionic states. For this purpose, we need the $[1]\times[21]\to[31]$ ORC, taken from Chen *et al* (1978a) and listed in table 1.

Table 1. The ORC $\langle[31]m|[1]m_1[21]m_2\rangle$.

$[31]m$ \ $[1]m_1[21]m_2$		N†	$1,\begin{smallmatrix}23\\4\end{smallmatrix}$	$2,\begin{smallmatrix}13\\4\end{smallmatrix}$	$3,\begin{smallmatrix}12\\4\end{smallmatrix}$	$4,\begin{smallmatrix}12\\3\end{smallmatrix}$	$1,\begin{smallmatrix}24\\3\end{smallmatrix}$	$2,\begin{smallmatrix}14\\3\end{smallmatrix}$	$3,\begin{smallmatrix}14\\2\end{smallmatrix}$	$4,\begin{smallmatrix}13\\2\end{smallmatrix}$
[31] 1	123 4	$\sqrt{3}$	1	1	1					
[31] 2	124 3	$\sqrt{96}$	−1	−1	2	6	$\sqrt{27}$	$\sqrt{27}$		
[31] 3	134 2	$\sqrt{32}$	−1	1			$-\sqrt{3}$	$\sqrt{3}$	$\sqrt{12}$	$\sqrt{12}$

† N is the norm.

From (4.9), table 1 in this paper as well as table 3 in Chen *et al* (1983b), one has

$$\left|\ \begin{smallmatrix}1 & \mathring{3} & 4\\ 2 & & \end{smallmatrix},\ \boxed{a}\ \begin{smallmatrix}\alpha & \alpha\\ \beta & \end{smallmatrix}\right\rangle$$

$$=\sum_{m_1 m_2}\langle[31]3|[1]m_1[21]m_2\rangle\Lambda_{r_2}\left|\begin{matrix}[1]\\ m_1,\end{matrix}\ \boxed{a}\right\rangle\left|\begin{matrix}[21]\\ \tilde{m}_2,\end{matrix}\ \begin{smallmatrix}\alpha & \alpha\\ \beta & \end{smallmatrix}\right\rangle$$

$$=\sqrt{\tfrac{1}{32}}\Bigg[|1,\ a\rangle\left|\begin{matrix}24 & \alpha\alpha\\ 3, & \beta\end{matrix}\right\rangle-|2,\ a\rangle\left|\begin{matrix}14 & \alpha\alpha\\ 3, & \beta\end{matrix}\right\rangle-\sqrt{3}|1,\ a\rangle\left|\begin{matrix}23 & \alpha\alpha\\ 4, & \beta\end{matrix}\right\rangle$$

$$+\sqrt{3}|2,\ a\rangle\left|\begin{matrix}13 & \alpha\alpha\\ 4, & \beta\end{matrix}\right\rangle+\sqrt{12}|3,a\rangle\left|\begin{matrix}12 & \alpha\alpha\\ 4, & \beta\end{matrix}\right\rangle+\sqrt{12}|4,\ a\rangle\left|\begin{matrix}12 & \alpha\alpha\\ 3, & \beta\end{matrix}\right\rangle\Bigg]$$

$$=\tfrac{1}{4}[-|a\alpha\alpha\beta\rangle+|a\alpha\beta\alpha\rangle+|\alpha a\alpha\beta\rangle-|\alpha a\beta\alpha\rangle+2|\alpha\alpha a\beta\rangle$$

$$-|\alpha\beta a\alpha\rangle-|\beta\alpha a\alpha\rangle+2|\alpha\alpha\beta a\rangle-|\alpha\beta\alpha a\rangle-|\beta\alpha\alpha a\rangle]. \tag{4.11}$$

5. The $U(m+p/n+q) \supset U(m/n) \times U(p/q)$ CFP

To simplify notation, let $M = n+p$ and $N = n+q$. The convenient bases for the subsystems $A^{f_1}B^{f_2}$ and $A^{f_3}B^{f_4}$ are the YB of $\mathring{S}(f_{12})$ and $U(M/N) \supset U(m/n) \times U(p/q)$ IRB, and the YB of $\mathring{S}(f_{34})$ and $U(M/N) \supset U(m/n) \times U(p/q)$ IRB, respectively, i.e.

$$\left|\begin{matrix}\sigma_{12} \\ m_{12}, \sigma_1 w_1 \sigma_2 w_2\end{matrix}\right\rangle^{\circ}, \qquad \left|\begin{matrix}\sigma_{34} \\ m_{34}, \sigma_3 w_3 \sigma_4 w_4\end{matrix}\right\rangle^{\circ}. \tag{5.1}$$

The basis vectors in (5.1) can be coupled (direct coupling!) to the $U(M/N)$ IRB by using the $U(M/N) \supset U(m/n) \times U(p/q)$ CGC, which can be factorised as

$$U(M/N) \supset U(m/n) \times U(p/q) \text{ CGC} = (U(M/N) \supset U(m/n) \times U(p/q) \text{ CFP})$$
$$\times U(m/n) \text{ CGC} \times U(p/q) \text{ CGC},$$

and the resulting basis, due to theorem 4, is also the $\mathring{S}(f) \supset \mathring{S}(f_{12}) \times \mathring{S}(f_{34})$ IRB, namely

$$\left|\begin{matrix}\sigma \\ \sigma_{12} m_{12} \sigma_{34} m_{34}, \sigma_{13} w_{13} \sigma_{24} w_{24}\end{matrix}\right\rangle^{\circ}$$
$$= \sum_{\sigma_1 \ldots \sigma_4} \left\langle\begin{matrix}\sigma \\ \sigma_{13}\sigma_{24}\end{matrix}\middle|\begin{matrix}\sigma_{12} & \sigma_{34} \\ \sigma_1\sigma_2, & \sigma_3\sigma_4\end{matrix}\right\rangle_{U(M/N)}$$
$$\times \left[\left|\begin{matrix}\sigma_{12} \\ m_{12}, \sigma_1\sigma_2\end{matrix}\right\rangle^{\circ} \left|\begin{matrix}\sigma_{34} \\ m_{34}, \sigma_3\sigma_4\end{matrix}\right\rangle^{\circ}\right]^{\sigma_{13}\sigma_{24}}_{w_{13}w_{24}}, \tag{5.2}$$

where the first factor on the RHS is the $U(M/N) \supset U(m/n) \times U(p/q)$ CFP. The square bracket denotes the coupling in terms of the $U(m/n)$ and $U(p/q)$ CGC simultaneously,

$$\mathscr{A} \equiv \left[\left|\begin{matrix}\sigma_{12} \\ m_{12}, \sigma_1\sigma_2\end{matrix}\right\rangle^{\circ} \left|\begin{matrix}\sigma_{34} \\ m_{34}, \sigma_3\sigma_4\end{matrix}\right\rangle^{\circ}\right]^{\sigma_{13}\sigma_{24}}_{w_{13}w_{24}}$$
$$\equiv \sum_{w_1 \ldots w_4} {}^{\circ}\langle \sigma_{13} w_{13} | \sigma_1 w_1 \sigma_3 w_3 \rangle^{\circ}\, {}^{\circ}\langle \sigma_{24} w_{24} | \sigma_2 w_2 \sigma_4 w_4 \rangle^{\circ}$$
$$\times \left|\begin{matrix}\sigma_{12} \\ m_{12}, \sigma_1 w_1 \sigma_2 w_2\end{matrix}\right\rangle^{\circ} \left|\begin{matrix}\sigma_{34} \\ m_{34}, \sigma_3 w_3 \sigma_4 w_4\end{matrix}\right\rangle^{\circ}. \tag{5.3a}$$

By using (3.3),

$$\mathscr{A} = \sum_{w_1 \ldots w_4} \sum_{m_1 \ldots m_4} {}^{\circ}\langle \sigma_{13} w_{13} | \sigma_1 w_1 \sigma_3 w_3 \rangle^{\circ}\, {}^{\circ}\langle \sigma_{24} w_{24} | \sigma_2 w_2 \sigma_4 w_4 \rangle^{\circ}$$
$$\times \langle \sigma_{12} m_{12} | \sigma_1 m_1 \sigma_2 m_2 \rangle \langle \sigma_{34} m_{34} | \sigma_3 m_3 \sigma_4 m_4 \rangle \prod_{i=1}^{4} \left|\begin{matrix}\sigma_i \\ m_i w_i\end{matrix}\right\rangle^{\circ}. \tag{5.3b}$$

The above procedure for constructing (5.2) is, so to speak, the unitary group approach. Alternatively, we can use the permutation group approach to construct the *same* basis. Out of the following two sets of basis vectors,

$\mathring{S}'(f_{13}) \supset \mathring{S}'(f_1) \times \mathring{S}'(f_3)$	$\mathring{S}'(f_{24}) \supset \mathring{S}'(f_2) \times \mathring{S}'(f_4)$
and $U(m/n)$ IRB	and $U(p/q)$ IRB

$$\left|\begin{matrix}\sigma_{13} \\ \sigma_1 m_1 \sigma_3 m_3, w_{13}\end{matrix}\right\rangle^{\circ}, \qquad \left|\begin{matrix}\sigma_{24} \\ \sigma_2 m_2 \sigma_4 m_4, w_{24}\end{matrix}\right\rangle^{\circ}, \tag{5.4}$$

by using the NORC (2.9a), we can construct the $\mathring{S}(f) \supset \mathring{S}(f_{12}) \times \mathring{S}(f_{34})$ IRB, which is, due to theorem 3, also the $U(M/N) \supset U(m/n) \times U(p/q)$ IRB. Thus we have

$$\left| \begin{matrix} \sigma \\ \sigma_{12} m_{12} \sigma_{34} m_{34}, \sigma_{13} w_{13} \sigma_{24} w_{24} \end{matrix} \right\rangle^{\circ} = \sum_{\sigma_1 \ldots \sigma_4} \left\langle \begin{matrix} \sigma \\ \sigma_{12}\sigma_{34} \end{matrix} \middle| \begin{matrix} \sigma_{13} & \sigma_{24} \\ \sigma_1\sigma_3, & \sigma_2\sigma_4 \end{matrix} \right\rangle_{S(f)} \times \left\{ \left| \begin{matrix} \sigma_{13} \\ \sigma_1\sigma_3, w_{13} \end{matrix} \right\rangle^{\circ} \left| \begin{matrix} \sigma_{24} \\ \sigma_2\sigma_4, w_{24} \end{matrix} \right\rangle^{\circ} \right\}^{\sigma_{12}\sigma_{34}}_{m_{12}m_{34}}, \tag{5.5a}$$

where the braces indicate that the bases are to be combined into the YB of $\mathring{S}(f_{12})$ and $\mathring{S}(f_{34})$ in terms of the ORC of $\mathring{S}(f_{12})$ and $\mathring{S}(f_{34})$, i.e.

$$\mathscr{B} \equiv \left\{ \left| \begin{matrix} \sigma_{13} \\ \sigma_1\sigma_3, w_{13} \end{matrix} \right\rangle^{\circ} \left| \begin{matrix} \sigma_{24} \\ \sigma_2\sigma_4, w_{24} \end{matrix} \right\rangle^{\circ} \right\}^{\sigma_{12}\sigma_{34}}_{m_{12}m_{34}} \equiv \sum_{m_1 \ldots m_4} \langle \sigma_{12} m_{12} | \sigma_1 m_1 \sigma_2 m_2 \rangle \langle \sigma_{34} m_{34} | \sigma_3 m_3 \sigma_4 m_4 \rangle \times \left| \begin{matrix} \sigma_{13} \\ \sigma_1 m_1 \sigma_3 m_3, w_{13} \end{matrix} \right\rangle^{\circ} \left| \begin{matrix} \sigma_{24} \\ \sigma_2 m_2 \sigma_4 m_4, w_{24} \end{matrix} \right\rangle^{\circ}. \tag{5.5b}$$

Using (3.4), it is easily seen that $\mathscr{B} = \mathscr{A}$. Thus the equality of the LHS of (5.2) and (5.5a) leads to the identification of the $U(M/N) \supset U(m/n) \times U(p/q)$ CFP with the $S(f) \supset S(f_{12}) \times S(f_{34})$ OISF,

$$\left\langle \begin{matrix} \sigma \\ \sigma_{13}\sigma_{24} \end{matrix} \middle| \begin{matrix} \sigma_{12} & \sigma_{34} \\ \sigma_1\sigma_2, & \sigma_3\sigma_4 \end{matrix} \right\rangle_{U(M/N)} = \left\langle \begin{matrix} \sigma \\ \sigma_{12}\sigma_{34} \end{matrix} \middle| \begin{matrix} \sigma_{13} & \sigma_{24} \\ \sigma_1\sigma_3, & \sigma_2\sigma_4 \end{matrix} \right\rangle_{S(f)}. \tag{5.6}$$

Since the OISF is independent of m, n, p and q, from (5.6) we know that the $U(m+p/n+q) \supset U(m/n) \times U(p/q)$ CFP is identical to the $U(m+p) \supset U(m) \times U(p)$ CFP. Combining (5.6) with (2.10), we see that the OISF, the $U(m+p) \supset U(m) \times U(p)$ and $U(m+p/n+q) \supset U(m/n) \times U(p/q)$ CFP are identical to one another, and have the symmetry

$$\left\langle \begin{matrix} \sigma \\ \sigma_{13}\sigma_{24} \end{matrix} \middle| \begin{matrix} \sigma_{12} & \sigma_{34} \\ \sigma_1\sigma_2, & \sigma_3\sigma_4 \end{matrix} \right\rangle^{\theta_{12}\theta_{34}\theta_{1234}}_{\theta_{13}\theta_{24}\theta_{1324}} = \left\langle \begin{matrix} \sigma \\ \sigma_{12}\sigma_{34} \end{matrix} \middle| \begin{matrix} \sigma_{13} & \sigma_{24} \\ \sigma_1\sigma_3, & \sigma_2\sigma_4 \end{matrix} \right\rangle^{\theta_{13}\theta_{24}\theta_{1324}}_{\theta_{12}\theta_{34}\theta_{1234}}, \tag{5.7}$$

where we have deleted the redundant subscripts $S(f)$, $U(m+n)$ and $U(M/N)$ in the ISF. From (2.12) and the symmetries of ORC and TC (Chen *et al* 1978a, Kramer 1968), we have another symmetry for the CFP,

$$\left\langle \begin{matrix} \sigma \\ \sigma_{13}\sigma_{24} \end{matrix} \middle| \begin{matrix} \sigma_{12} & \sigma_{34} \\ \sigma_1\sigma_2, & \sigma_3\sigma_4 \end{matrix} \right\rangle^{\theta_{12}\theta_{34}\theta_{1234}}_{\theta_{13}\theta_{24}\theta_{1324}} = \eta \left\langle \begin{matrix} \tilde{\sigma} \\ \tilde{\sigma}_{13}\tilde{\sigma}_{24} \end{matrix} \middle| \begin{matrix} \tilde{\sigma}_{12} & \tilde{\sigma}_{34} \\ \tilde{\sigma}_1\tilde{\sigma}_2, & \tilde{\sigma}_3\tilde{\sigma}_4 \end{matrix} \right\rangle^{\theta_{12}\theta_{34}\theta_{1234}}_{\theta_{13}\theta_{24}\theta_{1324}} \tag{5.8}$$

where η is a phase factor depending on all the nine partitions.

It should be stressed that in (2.10) and (5.6), there is no extra adjustable phase factor between the OISF and CFP, once the YY and GZ phase conventions are used for reps of $S(f)$, $U(m)$ and $U(m/n)$ (Chen and Chen 1983).

6. The $U(m/n) \supset U(m)\times U(n)$ CFP

Suppose that $n = p = 0$; then the IRB $|{}^{\sigma_i}_{m_i w_i}\rangle^\circ$ in (3.2) are bosonic for $i = 1, 3$ and fermionic for $i = 2, 4$. In (4.9) we showed how to construct a $U(m/q) \supset U(m)\times U(q)$ IRB for the subsystem $A^{f_1}B^{f_2}$. A similar equation exists for the subsystem $A^{f_3}B^{f_4}$. Similar to (5.2), the CFP expansion for the $U(m/q)\supset U(m)\times U(q)$ IRB of the total system is as follows:

$$\left|\begin{matrix} \mathring{\sigma} \\ \mathring{\sigma}_{12}\mathring{m}_{12}\mathring{\sigma}_{34}\mathring{m}_{34},\ \sigma_{13}w_{13}\sigma_{24}w_{24}\end{matrix}\right\rangle = \sum_{\sigma_1\ldots\sigma_4}\left(\begin{matrix}\sigma \\ \sigma_{13}\sigma_{24}\end{matrix}\middle|\begin{matrix}\sigma_{12} & \sigma_{34} \\ \sigma_1\sigma_2, & \sigma_3\sigma_4\end{matrix}\right)\left[\left|\begin{matrix}\mathring{\sigma}_{12} \\ \mathring{m}_{12},\ \sigma_1\sigma_2\end{matrix}\right\rangle\left|\begin{matrix}\mathring{\sigma}_{34} \\ \mathring{m}_{34},\ \sigma_3\sigma_4\end{matrix}\right\rangle\right]^{\sigma_{13}\sigma_{24}}_{w_{13}w_{24}}, \tag{6.1}$$

where the first factor on the RHS is the $U(m/q)\supset U(m)\times U(q)$ f_{34}-body CFP, and the square brackets denote coupling in terms of the CGC of $U(m)$ and $U(q)$, i.e.

$$\begin{aligned}\mathscr{A}' &= \left[\left|\begin{matrix}\mathring{\sigma}_{12} \\ \mathring{m}_{12},\ \sigma_1\sigma_2\end{matrix}\right\rangle\left|\begin{matrix}\mathring{\sigma}_{34} \\ \mathring{m}_{34},\ \sigma_3\sigma_4\end{matrix}\right\rangle\right]^{\sigma_{13}\sigma_{24}}_{w_{13}w_{24}} \\ &= \sum_{w_1\ldots w_4}\langle\sigma_{13}w_{13}|\sigma_1 w_1\sigma_3 w_3\rangle\langle\sigma_{24}w_{24}|\sigma_2 w_2\sigma_4 w_4\rangle \\ &\quad\times\left|\begin{matrix}\mathring{\sigma}_{12} \\ \mathring{m}_{12},\ \sigma_1 w_1\sigma_2 w_2\end{matrix}\right\rangle\left|\begin{matrix}\mathring{\sigma}_{34} \\ \mathring{m}_{34},\ \sigma_3 w_3\sigma_4 w_4\end{matrix}\right\rangle.\end{aligned} \tag{6.2a}$$

By (4.9), we have

$$\begin{aligned}\mathscr{A}' = \sum_{w_1\ldots w_4}\sum_{m_1\ldots m_4}&\langle\sigma_{13}w_{13}|\sigma_1 w_1\sigma_3 w_3\rangle\langle\sigma_{24}w_{24}|\sigma_2 w_2\sigma_4 w_4\rangle \\ &\times\langle\sigma_{12}m_{12}|\sigma_1 m_1\tilde{\sigma}_2\tilde{m}_2\rangle\Lambda_{\tilde{f}_2}\langle\sigma_{34}m_{34}|\sigma_3 m_3\tilde{\sigma}_4\tilde{m}_4\rangle\Lambda_{\tilde{f}_4}\prod_{i=1}^{4}\left|\begin{matrix}\sigma_i \\ m_i w_i\end{matrix}\right\rangle.\end{aligned} \tag{6.2b}$$

The above procedure is the unitary group approach. On the other hand, from (4.9) and (5.5a) we can get the same basis vector as (6.1) via the permutation group approach,

$$\begin{aligned}\left|\begin{matrix} \mathring{\sigma} \\ \mathring{\sigma}_{12}\mathring{m}_{12}\mathring{\sigma}_{34}\mathring{m}_{34},\ \sigma_{13}w_{13}\sigma_{24}w_{24}\end{matrix}\right\rangle &= \Lambda_{\tilde{w}_{24}}\left|\begin{matrix}\sigma \\ \sigma_{12}m_{12}\sigma_{34}m_{34},\ \sigma_{13}w_{13}\tilde{\sigma}_{24}\tilde{w}_{24}\end{matrix}\right\rangle^\circ \\ &= \sum_{w_1\ldots w_4}\left\langle\begin{matrix}\sigma \\ \sigma_{12}\sigma_{34}\end{matrix}\middle|\begin{matrix}\sigma_{13} & \tilde{\sigma}_{24} \\ \sigma_1\sigma_3, & \tilde{\sigma}_2\tilde{\sigma}_4\end{matrix}\right\rangle\mathscr{B}'\end{aligned} \tag{6.3}$$

where

$$\begin{aligned}\mathscr{B}' &\equiv \Lambda_{\tilde{w}_{24}}\left\{\left|\begin{matrix}\sigma_{13} \\ \sigma_1\sigma_3,\ w_{13}\end{matrix}\right\rangle^\circ\left|\begin{matrix}\tilde{\sigma}_{24} \\ \tilde{\sigma}_2\tilde{\sigma}_4,\ \tilde{w}_{24}\end{matrix}\right\rangle^\circ\right\}^{\sigma_{12}\sigma_{34}}_{m_{12}m_{34}} \\ &\equiv \Lambda_{\tilde{w}_{24}}\sum_{m_1\ldots m_4}\langle\sigma_{12}m_{12}|\sigma_1 m_1\tilde{\sigma}_2\tilde{m}_2\rangle\langle\sigma_{34}m_{34}|\sigma_3 m_3\tilde{\sigma}_4\tilde{m}_4\rangle \\ &\quad\times\left|\begin{matrix}\sigma_{13} \\ \sigma_1 m_1\sigma_3 m_3,\ w_{13}\end{matrix}\right\rangle\left|\begin{matrix}\tilde{\sigma}_{24} \\ \tilde{\sigma}_2\tilde{m}_2\tilde{\sigma}_4\tilde{m}_4,\ \tilde{w}_{24}\end{matrix}\right\rangle^\circ.\end{aligned} \tag{6.4}$$

By transforming the non-standard basis of $\mathring{S}(f_{24})$ to the standard one, and making use of (4.7a) as well as the symmetry of the TC of $S(f_{24})$ under the conjugation, we obtain

$$\left|\begin{matrix}\tilde\sigma_{24}\\ \tilde\sigma_2\tilde m_2\tilde\sigma_4\tilde m_4,\ \tilde w_{24}\end{matrix}\right\rangle^{\circ}=\varepsilon(\tilde\sigma_2\tilde\sigma_4\tilde\sigma_{24})\Lambda_{\tilde w_{24}}\Lambda_{\tilde r_2}\Lambda_{\tilde r_4}\left|\begin{matrix}\sigma_{24}\\ \sigma_2 m_2\sigma_4 m_4,\ w_{24}\end{matrix}\right\rangle, \tag{6.5}$$

where $\varepsilon(\tilde\sigma_2\tilde\sigma_4\tilde\sigma_{24})$ is a phase factor determined by the absolute phase convention of the TC (Chen *et al* 1983c).

Substituting (6.5) into (6.4), and using (3.4) (which is still valid for $n=0$), we can prove that

$$\mathscr{B}'=\varepsilon(\tilde\sigma_2\tilde\sigma_4\tilde\sigma_{24})\mathscr{A}'. \tag{6.6}$$

On equating the LHS of (6.1) with (6.3) and using (5.7), we obtain a very simple relation

$$\left(\begin{matrix}\sigma\\ \sigma_{13}\sigma_{24}\end{matrix}\right|\left.\begin{matrix}\sigma_{12} & \sigma_{34}\\ \sigma_1\sigma_2, & \sigma_3\sigma_4\end{matrix}\right)=\varepsilon(\tilde\sigma_2\tilde\sigma_4\tilde\sigma_{24})\left\langle\begin{matrix}\sigma\\ \sigma_{13}\tilde\sigma_{24}\end{matrix}\right|\left.\begin{matrix}\sigma_{12} & \sigma_{34}\\ \sigma_1\tilde\sigma_2, & \sigma_3\tilde\sigma_4\end{matrix}\right\rangle, \tag{6.7}$$

namely, the $U(m/n)\supset U(m)\times U(n)$ CFP and $U(m+n)\supset U(m)\times U(n)$ CFP differ only in the labelling for the irreps of $U(n)$ and a phase factor $\varepsilon(\tilde\sigma_2\tilde\sigma_4\tilde\sigma_{24})$, which are easily determined from the TC tables and have been listed in table 1.3 in Chen *et al* (1983c). A relation between $\varepsilon(\sigma_2\sigma_4\sigma_{24})$ and $\varepsilon(\tilde\sigma_2\tilde\sigma_4\tilde\sigma_{24})$ is given by equation (4.5) in Kramer (1968).

For the one-body CFP, $f_{34}=\sigma_{34}=1$, the phase factors $\varepsilon(\tilde\sigma_2\tilde\sigma_4\tilde\sigma_{24})$ with $\sigma_4=1$ or 0 are identically equal to one. Again using the simplified notations for the partitions shown in (2.13a), from (5.7) and (6.7) we obtain relations about the one-body CFP as follows:

$$\left(\begin{matrix}\sigma\\ \sigma_1\sigma_2\end{matrix}\right|\left.\begin{matrix}\sigma' & [1]\\ \sigma_1'\sigma_2', & \end{matrix}\right)^{\theta}_{\theta'}=\left\langle\begin{matrix}\sigma\\ \sigma_1\tilde\sigma_2\end{matrix}\right|\left.\begin{matrix}\sigma' & [1]\\ \sigma_1'\tilde\sigma_2', & \end{matrix}\right\rangle^{\theta}_{\theta'}=\left\langle\begin{matrix}\sigma\\ \sigma'\end{matrix}\right|\left.\begin{matrix}\sigma_1 & \tilde\sigma_2\\ \sigma_1' & \tilde\sigma_2'\end{matrix}\right\rangle^{\theta}_{\theta'}. \tag{6.8}$$

It is thus seen that the $U(m/n)\supset U(m)\times U(n)$ and $U(m+n)\supset U(m)\times U(n)$ one-body CFP can be listed in the same table, as is done in § 7.

Special case. For a general $U(m/n)$ or $U(m+n)$ CFP, it is not possible to get closed analytic formulae, and one has to be content with numerical results. Nevertheless, simple expressions do exist for the special cases where the irrep $[\sigma]$ is either totally symmetric or totally antisymmetric. Let us first consider the $U(m+n)$ case with totally symmetric rep $[\sigma]=[f]$. In such a case, all the irreps $[\sigma_i]$ are necessarily also totally symmetric,

$$[\sigma_i]=[f_i], \qquad i=1,\ldots,4,12,34,13,24.$$

The NORC in (2.8) reduces to the standard ORC for the totally symmetric reps, which is just equal to the binomial coefficient (Chen 1984),

$$\left\langle\begin{matrix}\sigma\\ \sigma_{12}m_{12}\sigma_{34}m_{34}\end{matrix}\right|\left.\begin{matrix}\sigma_{13} & \sigma_{24}\\ \sigma_1 m_1\sigma_3 m_3, & \sigma_2 m_2\sigma_4 m_4\end{matrix}\right\rangle$$
$$=\langle[f]m|[f_{13}]m_{13},[f_{24}]m_{24}\rangle=(f_{13}!f_{24}!/f!)^{1/2}. \tag{6.9}$$

From (2.9a) and (6.9) we obtain a simple expression for the $U(m+n)\supset U(m)\times U(n)$ CFP

$$\left\langle\begin{matrix}[f]\\ [f_{12}][f_{34}]\end{matrix}\right|\left.\begin{matrix}[f_{13}] & [f_{24}]\\ [f_1][f_3], & [f_2][f_4]\end{matrix}\right\rangle=\left(\frac{f_{12}!f_{34}!f_{13}!f_{24}!}{f_1!f_2!f_3!f_4!f!}\right)^{1/2}. \tag{6.10}$$

The $U(m+n) \supset U(m) \times U(n)$ CFP for the totally antisymmetric irrep $[\sigma]=[\tilde{f}]$ is obtained from (6.10) through the use of the symmetry relation (5.8).

Using (6.9), we immediately obtain the $U(m/n) \supset U(m) \times U(n)$ CFP for the totally symmetric irrep $[\sigma]=[f]$ of $U(m/n)$,

$$\left(\begin{array}{c|cc} [f] & [f_{12}] & [f_{34}] \\ [f_{13}][\tilde{f}_{24}] & [f_1][\tilde{f}_2], & [f_3][\tilde{f}_4] \end{array}\right) = \left(\frac{f_{12}!\, f_{34}!\, f_{13}!\, f_{24}!}{f_1!\, f_2!\, f_3!\, f_4!\, f!}\right)^{1/2}, \tag{6.11}$$

with $[f_1]$, $[f_3]$ and $[f_{13}]$ specifying the totally symmetric representations of the boson subsystems, and $[\tilde{f}_2]$, $[\tilde{f}_4]$ and $[\tilde{f}_{24}]$ specifying the totally antisymmetric representations of the fermion subsystems. The special CFP of (6.11) is of importance in the study of the supersymmetry in nuclei with the interacting boson model (Balantekin *et al* 1981).

7. Tables of the one-body CFP

With the ORC table (Chen and Gao 1981) and equation (2.13), the one-body CFP for $U(m+n) \supset U(m) \times U(n)$ or $U(m/n) \supset U(m) \times U(n)$ have been calculated for systems with up to six particles. The results are given in tables 2–26. All the partitions are arranged in order of decreasing row symmetry from top to bottom in the corresponding Young diagrams. The tables are arranged in the order of $([\sigma_1][\sigma_2][\sigma'])$. We only listed the CFP for the products $[\sigma_1] \times [\sigma_2]$, where $[\sigma_2]$ is below $[\sigma_1]$ and $[\sigma_1]$ is no lower than the self-conjugate partition. The remaining CFP can be found from the symmetries of the CFP. For example

$$\left\langle\begin{array}{c|cc} [\sigma] & [\sigma'] & [1] \\ \sigma_1\sigma_2 & \sigma'_1\sigma'_2, & \end{array}\right\rangle^{\theta}_{\theta'} = \eta_1 \left\langle\begin{array}{c|cc} [\sigma] & [\sigma'] & [1] \\ \sigma_2\sigma_1 & \sigma'_2\sigma'_1, & \end{array}\right\rangle^{\theta}_{\theta'},$$
$$\eta_1 = \varepsilon_1(\sigma_1\sigma_2\sigma\theta)\,\varepsilon_1(\sigma'_1\sigma'_2\sigma'\theta'), \tag{7.1}$$

where the phase factors $\varepsilon_1(\sigma_1\sigma_2\sigma\theta)$ and $\varepsilon_1(\sigma'_1\sigma'_2\sigma'\theta')$ come from the symmetry of the ORC (see equation (4-153) in Chen 1984) and are listed in table A2 in Chen (1984).

The table headings have the following meaning:

$[\sigma']$	$[\sigma_1] \times [\sigma_2]$ $[\sigma]\theta$
$[\sigma'_1][\sigma'_2]\theta'$	

All the entries represent the square values of the CFP, and a minus entry signifies a negative CFP value.

A whole class of unit CFP has not been included in the tables. Whether a CFP is a unit CFP can be judged from the Littlewood rule

$$[\sigma_1] \times [\sigma_2] = \sum_{\sigma} \oplus \{\sigma_1\sigma_2\sigma\}[\sigma]. \tag{7.2}$$

Tables for the multiplicity $\{\sigma_1\sigma_2\sigma\}$ are available (Itzykson and Nauenberg 1966). We have

$$\left\langle\begin{array}{c|cc} [\sigma] & [\sigma'] & [1] \\ \sigma_1\sigma_2 & \sigma'_1\sigma'_2, & \end{array}\right\rangle = 1 \qquad \text{if } \{\sigma_1\sigma_2\sigma\} = \{\sigma'_1\sigma'_2\sigma'\} = 1. \tag{7.3}$$

Tables 2–26. The one-body CFP

$$\left\langle \begin{matrix} [\sigma] \\ \sigma_1\sigma_2 \end{matrix} \middle| \begin{matrix} [\sigma'] & [1] \\ \sigma_1'\sigma_2', & \end{matrix} \right\rangle_{\theta'}^{\theta}$$

for $U(m+n) \supset U(m) \times U(n)$ or $U(m+p/n+q) \supset U(m/n) \times U(p/q)$,

and the one-body CFP

$$\left(\begin{matrix} [\sigma] \\ \sigma_1\sigma_2 \end{matrix} \middle| \begin{matrix} [\sigma'] & [1] \\ \sigma_1'\tilde{\sigma}_2', & \end{matrix} \right)_{\theta'}^{\theta}$$ for $U(m/n) \supset U(m) \times U(n)$.

Table 2. (2)×(1)

(2)	(2)×(1) (3)	(21)
(2) (0)	1/3	2/3
(1) (1)	2/3	−1/3

Table 3. (3)×(1)

(3)	(3)×(1) (4)	(31)
(3) (0)	1/4	3/4
(2) (1)	3/4	−1/4

Table 4. (21)×(1)

(21)	(21)×(1) (31)	(22)	(211)
(21) (0)	3/8	1/4	3/8
(2) (1)	1/16	3/8	−9/16
(11) (1)	9/16	−3/8	−1/16

Table 5. (*a*) (2)×(2)

(3)	(2)×(2) (4)	(31)
(2) (1)	1/2	1/2
(1) (2)	1/2	−1/2

Table 5. (*b*)

(21)	(2)×(2) (31)	(22)
(2) (1)	1/2	1/2
(1) (2)	−1/2	1/2

Table 6. (2)×(11)

(21)	(2)×(11) (31)	(211)
(2) (1)	−1/4	3/4
(1) (11)	3/4	1/4

Table 7. (4)×(1)

(4)	(4)×(1) (5)	(41)
(4) (0)	1/5	4/5
(3) (1)	4/5	−1/5

Table 8. (31)×(1)

(31)	(31)×(1) (41)	(32)	(311)
(31) (0)	4/15	1/3	2/5
(3) (1)	1/45	4/9	−8/15
(21) (1)	32/45	−2/9	−1/15

Table 9. (22)×(1)

(22)	(22)×(1) (32)	(221)
(22) (0)	1/2	1/2
(21) (1)	1/2	−1/2

Table 10. (211)×(1)

(211)	(211)×(1) (311)	(221)	(21^3)
(211) (0)	2/5	1/3	4/15
(21) (1)	1/15	2/9	−32/45
(111) (1)	8/15	−4/9	−1/45

Table 11. (*a*) (3)×(2)

(4)	(3)×(2) (5)	(41)
(3) (1)	2/5	3/5
(2) (2)	3/5	−2/5

Table 11. (*b*)

(31)	(3)×(2) (41)	(32)
(3) (1)	1/3	2/3
(2) (2)	2/3	−1/3

Table 12. (3)×(11)

(31)	(3)×(11) (41)	(311)
(3) (1)	−1/5	4/5
(2) (11)	4/5	1/5

Table 13. (*a*) (21)×(2)

(31)	(21)×(2) (41)	(32)	(311)
(21) (1)	8/15	1/6	3/10
(2) (2)	1/15	1/3	−3/5
(11) (2)	2/5	−1/2	−1/10

Table 13. (*b*)

(22)	(21)×(2) (32)	(221)
(21) (1)	3/4	1/4
(2) (2)	1/4	−3/4

Table 13. (*c*)

(211)	(21)×(2) (311)	(221)
(21) (1)	1/2	1/2
(11) (2)	1/2	−1/2

Table 14. (*a*) (21)×(11)

(31)	(21)×(11) (32)	(311)
(21) (1)	1/2	1/2
(2) (11)	1/2	−1/2

Table 14. (*b*)

(22)	(21)×(11) (32)	(221)
(21) (1)	−1/4	3/4
(11) (11)	3/4	1/4

Table 14. (*c*)

(211)	(21)×(11) (311)	(221)	(21^3)
(21) (1)	−3/10	−1/6	8/15
(2) (11)	1/10	1/2	2/5
(11) (11)	3/5	−1/3	1/15

Table 15. (5)×(1)

(5)	(5)×(1) (6)	(51)
(5) (0)	1/6	5/6
(4) (1)	5/6	−1/6

Table 16. (41)×(1)

(41)	(41)×(1) (51)	(42)	(411)
(41) (0)	5/24	3/8	5/12
(4) (0)	1/96	15/32	−25/48
(31) (1)	25/32	−5/32	−1/16

Table 17. (32)×(1)

(32)	(32)×(1) (42)	(33)	(321)
(32) (0)	3/10	1/6	8/15
(31) (1)	1/10	1/2	−2/5
(22) (1)	3/5	−1/3	−1/15

Table 18. (311)×(1)

(311)	(311)×(1) (411)	(321)	(31^3)
(311) (0)	5/18	4/9	5/18
(31) (1)	1/36	5/18	−25/36
(211) (1)	25/36	−5/18	−1/36

Table 19. (221)×(1)

(221)	(221)×(1) (321)	(2^3)	(2^21^2)
(221) (0)	8/15	1/6	3/10
(22) (1)	1/15	1/3	−3/5
(211) (1)	2/5	−1/2	−1/10

Table 20. (*a*) (4)×(2)

(5)	(4)×(2) (6)	(51)
(4) (1)	1/3	2/3
(3) (2)	2/3	−1/3

Table 20. (*b*)

(41)	(4)×(2) (51)	(42)
(4) (1)	1/4	3/4
(3) (2)	3/4	−1/4

Table 21. (*a*) (31)×(2)

(41)	(31)×(2) (51)	(42)	(411)
(31) (1)	5/12	1/4	1/3
(3) (2)	1/36	5/12	−5/9
(21) (2)	5/9	−1/3	−1/9

Table 21. (*b*)

(32)	(31)×(2) (42)	(33)	(321)
(31) (1)	2/5	1/3	4/15
(3) (2)	1/15	2/9	−32/45
(21) (2)	8/15	−4/9	−1/45

Table 21. (*c*)

(311)	(31)×(2) (411)	(321)
(31) (1)	1/3	2/3
(21) (2)	2/3	−1/3

Table 22. (*a*) (22)×(2)

(32)	(22)×(2) (42)	(321)
(22) (1)	3/5	2/5
(21) (2)	2/5	−3/5

Table 22. (*b*)

(221)	(22)×(2) (321)	(2^3)
(22) (1)	2/3	1/3
(21) (2)	1/3	−2/3

Table 23. (*a*) (211)×(2)

(311)	(211)×(2) (411)	(321)	(31³)
(211) (1)	5/9	2/9	2/9
(21) (2)	2/27	5/27	−20/27
(1³) (2)	10/27	−16/27	−1/27

Table 23. (*b*)

(221)	(211)×(2) (321)	(2²1²)
(211) (1)	4/5	1/5
(21) (2)	1/5	−4/5

Table 23. (*c*)

(21³)	(211)×(2) (31³)	(2²1²)
(211) (1)	1/2	1/2
(1³) (2)	1/2	−1/2

Table 24. (*a*) (3)×(3)

(5)	(3)×(3) (6)	(51)
(3) (2)	1/2	1/2
(2) (3)	1/2	−1/2

Table 24. (*b*)

(41)	(3)×(3) (51)	(42)
(3) (2)	1/2	1/2
(2) (3)	1/2	−1/2

Table 24. (*c*)

(32)	(3)×(3) (42)	(33)
(3) (2)	1/2	1/2
(2) (3)	1/2	−1/2

Table 25. (*a*) (3)×(21)

(41)	(3)×(21) (51)	(42)	(411)
(3) (2)	1/16	5/16	−5/8
(3) (11)	−5/16	9/16	1/8
(2) (21)	−5/8	−1/8	−1/4

Table 25. (*b*)

(32)	(3)×(21) (42)	(321)
(3) (2)	−1/5	4/5
(2) (21)	4/5	1/5

Table 25. (*c*)

(311)	(3)×(21) (411)	(321)
(3) (11)	−1/3	2/3
(2) (21)	−2/3	−1/3

Table 26. (*a*) (21)×(21)

(41)	(21)×(21) (42)	(411)
(21) (2)	1/2	1/2
(2) (21)	1/2	−1/2

Table 26. (*b*)

(32)	(21)×(21) (42)	(321)α	(321)β	(33)
(21) (2)	−1/20	9/20	1/4	1/4
(21) (11)	9/20	1/20	−1/4	1/4
(2) (21)	−1/20	9/20	−1/4	−1/4
(11) (21)	9/20	1/20	1/4	−1/4

Table 26. (*c*)

(311)	(21)×(21) (411)	(312)α	(321)β	(31³)
(21) (2)	−1/12	1/12	−5/12	5/12
(21) (11)	5/12	5/12	−1/12	−1/12
(2) (21)	1/12	1/12	5/12	5/12
(11) (21)	−5/12	5/12	1/12	−1/12

Table 26. (*d*)

(221)	(21)×(21) (321)α	(321)β	(2³)	(2²1²)
(21) (2)	−1/4	1/20	1/4	9/20
(21) (11)	1/4	9/20	1/4	−1/20
(2) (21)	−1/4	−1/20	1/4	−9/20
(11) (21)	1/4	−9/20	1/4	1/20

8. Summary and discussions

The relations established by Kramer and Sullivan exist only between the invariants (recoupling coefficients) of U(n) and S(f) and no simple relation between the CGC of U(n) and the reduction coefficient of S(f) was known to exist before 1978. However, there must exist a relation between these two coefficients, since the $6f$ or $9f$ recoupling coefficient can be expressed either in terms of the CGC of $U(n)$, or the TC of $S(f)$. Equation (23) in Chen *et al* (1978a), or its extension, equation (3.4) in Chen *et al* (1984*a*), first gave a quantitative relation between the CGC of $U(n)$ and the outer-product reduction coefficient of $S(f)$. It was shown (Chen 1979, 1984) that the ORC and TC of $S(f)$ are related by

$$\langle\sigma r|\sigma_1 r_1\omega_1, \sigma_2 r_2\omega_2\rangle = C(\sigma_1\sigma_2\sigma)\sum_s \langle\sigma r|Q_\omega|\sigma s\rangle\langle\sigma s|\sigma_1 r_1\sigma_2 r_2\rangle$$

where $\langle\sigma r|Q_\omega|\sigma s\rangle$ are the Yamanouchi matrix elements for the order-preserving permutation Q_ω. Thus a direct relation between the CGC of $U(n)$ and the TC of $S(f)$ was established. A totally equivalent formula was derived by Nikam *et al* (1983, equation (18)) for the CGC of $U(n)$ in terms of the TC of $S(f)$.

Specialised to the $m=n=1$ case, the $U(m+n)\supset U(m)\times U(n)$ CFP is just the SU(2) CGC, and (2.11*a*) reduces to equation (3.8) in Kramer and Seligman (1969a), in which the SU(2) CGC is expressed in terms of the $9f$ recoupling coefficient of $S(f)$ in order to explain the Regge symmetry of the $3j$ symbol of SU(2).

From the present paper and our previous paper (Chen *et al* 1984a, b), we have reached the dualities shown in table 27 between the inner product and outer product of the permutation group on the one hand, and between the unitary group subduction $U(mn)\downarrow U(m)\times U(n)$ and $U(m+n)\downarrow U(m)\times U(n)$ on the other hand.

The branching rules for the subductions $SU(mn)\downarrow SU(m)\times SU(n)$ and $SU(m+n)\downarrow SU(m)\times SU(n)$ (Itzykson and Nauenberg 1966), as well as for $SU(mp+nq/mq+np)\downarrow SU(m/n)\times SU(p/q)$, $SU(m+p/n+q)\downarrow SU(m/n)\times SU(p/q)$ and $SU(m/n)\downarrow SU(m)\times SU(n)$ (Dondi and Jarvis 1981, Balantekin 1982) are the natural consequences of the dualities shown in table 27. These dualities not only greatly deepen our understanding of the permutation and unitary groups, but also

Table 27. Dualities between the unitary and permutation groups.

Permutation group	Unitary group
CGC (or the inner-product reduction coefficient) of $S(f)$	Indirect coupling coefficient for $U(mp+nq/mq+np)\supset U(m/n)\times U(p/q)$ IRB, or its special case, $U(mn)\supset U(m)\times U(n)$ IRB
ORC (the outer-product reduction coefficient) of $S(f)$	Indirect coupling coefficient for $U(m+p/n+q)\supset U(m/n)\times U(p/q)$ IRB, or its special case, $U(m+n)\supset U(m)\times U(n)$ IRB, or $U(m/n)\supset U(m)\times U(n)$ IRB†
ORC of $S(f)$	CGC for special GB of $U(m)$
ORC of $\mathring{\mathcal{S}}(f)$	CGC for special GB of $U(m/n)$
NORC of $S(f)$	CGC of $U(m+n)\supset U(m)\times U(n)$ for $U(m)$ and $U(n)$ special GB
Indirect coupling coefficient for $\hat{S}(f)\supset\hat{S}(f_1)\times\hat{S}(f_2)$	CGC of $U(m/n)$
$S(f)\supset S(f_1)\times S(f_2)$ inner-product ISF	f_2-body CFP for $U(mp+nq/mq+np)\supset U(m/n)\times U(p/q)$, or its special case, $U(mn)\supset U(m)\times U(n)$
$S(f)\supset S(f_{12})\times S(f_{34})$ outer-product ISF	f_{34}-body CFP for $U(m+p/n+q)\supset U(m/n)\times U(p/q)$, or its special case, $U(m+n)\supset U(m)\times U(n)$, or $U(m/n)\supset U(m)\times U(n)$†

† Due modifications are required in labelling and/or phase.

provide practical and rank-independent calculation methods for various kinds of unitary group CFP.

The last issue to be discussed is the phase problem. Since irreps of U(m) remain irreducible on restriction to its subgroup SU(m), the question arises as to whether the CGC or CFP may be the same for both groups. Bickerstaff and Damhus (1983) studied this problem in detail. Here we only mention that this is possible if appropriate phase choices are made for the U(m) coefficients. It is shown that the U(m) CGC evaluated from the ORC obey the Baird–Biedenharn phase convention (1965), and are identical with the SU(m) CGC (Chen *et al* 1984a). Since the phase of the CFP is decided by the phases of the CGC, we expect that the $U(m+n)\supset U(m)\times U(n)$ CFP tabulated in § 7 are also the $SU(m+n)\supset SU(m)\times SU(n)$ CFP. However, the problem is not clear yet with regard to the graded unitary group $U(m/n)$ and its subgroup $SU(m/n)$ and deserves further study.

Note added in proof. From (6.11) and the unitarity of the CFP, we can obtain an analytic expression for the $U(m/n)\supset U(m)\times U(n)$ one-body CFP for a system with N bosons and one fermion as shown in table 28.

Table 28. The $U(m/n)\supset U(m)\times U(1)$ CFP$\left(\begin{smallmatrix}[\sigma]\\ \sigma_1\sigma_2\end{smallmatrix}\middle|\begin{smallmatrix}[N][1]\\ \sigma_1'\sigma_2'\end{smallmatrix}\right)$.

	$[N]\times[1]$	
$[N]$	$[N+1]$	$[N,1]$
$[N][0]$	$\sqrt{\frac{1}{N+1}}$	$\sqrt{\frac{N}{N+1}}$
$[N-1][1]$	$\sqrt{\frac{N}{N+1}}$	$-\sqrt{\frac{1}{N+1}}$

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